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# A numerical boundary integral equation analysis for standard linear viscoelastic media made of functionally graded materials

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## Abstract

**Background:** Although structures made of functionally graded materials have been studied by many researchers, no research may be found in literature on boundary element analysis of the functionally graded viscoelastic structures.

**Methods:** In the present paper, a 2D boundary element formulation capable of modeling time-dependent functionally graded materials (FGM) is presented. A numerical implementation of the Somigliana identity in terms of the displacements is developed to solve 2D problems of the exponentially graded viscoelasticity. The FGM concept can be applied to various materials, for structural and functional purposes. In this model, only Green functions of the nonhomogeneous elastostatic problems are needed with material properties that vary continuously along a given dimension.

**Results and Conclusions:** Results reveal that the boundary element approach can successfully be employed for the present complicated problem for arbitrary time histories of the applied loads and arbitrary boundary conditions, without the need to use relaxation functions or mathematical transformations.

**Keywords:** Boundary integral equation formulation; Viscoelastic; FGMs; Nonhomogeneity; Time-dependent materials

## Background

In recent years, the functionally graded materials (FGMs), as a category of the composite materials, have generated a great deal of attention. An FGM is an advanced material whose composition changes gradually and results in corresponding changes in its properties (Suresh and Mortensen 1998). Because of the special features of the FGMs with potential applications to lots of engineering fields, they have attracted attention of numerous scientists and engineers in broad areas of research. Moreover, a variety of nonhomogeneous engineering media made of polymers, plastics, metals and alloys at elevated temperatures, composites, concrete, etc., exhibit significant rate and history dependencies. Appropriate simulation of these types of structures requires using appropriate viscoelastic models.

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Boundary element method (BEM) has recently found considerable applications in solving lots of engineering problems, such as viscoelasticity, contact mechanics, elastoplasticity, thermoelasticity, fracture mechanics, elastodynamics, etc. (Aliabadi 2002). The viscoelastic media can effectively and accurately be treated by the BEM (Ashrafi and Farid 2009; Ashrafi et al. 2012). The BEM just requires the boundary data as input, and there is no need for discretizing the domain under consideration into elements. Adaptation of the BEM to nonhomogeneous media is a hard task; because determination of the fundamental solutions corresponding to the concentrated loads is difficult for such materials. The fundamental solutions for heat transfer problems in the nonhomogeneous media have been presented using BEM algorithms by some researchers (Shaw and Makris 1992; Clements 1998; Clements and Budhi 1999; Gray et al. 2003; Sutradhar and Paulino 2004; Kuo and Chen 2005). Also, the fundamental solutions of the FGMs for 2D and 3D elasticity problems have been recently developed in some other works (Chan et al. 2004;

Criado et al. 2007, 2008). Using a Fourier transform technique, these functions have been derived for exponentially graded media. Although extensive works have been developed to effectively model the constitutive behavior of the FG structures, most of these researches have been limited to elastic behavior analysis (Gao et al. 2008; Wang and Qin 2012; Ashrafi et al. 2013a, 2013b).

## Methods

In this paper, the boundary element formulation is proposed based on the differential viscoelastic constitutive equations of nonhomogeneous SLS model. The resulting algorithm is capable of solving the quasistatic problems for exponentially graded viscoelastic materials with arbitrary boundary conditions and therefore, provides a reasonable model for certain realistic situations. Avoiding internal domain elements is one of the objectives of this paper, which results in numerical discretization of the boundary of the considered nonhomogeneous problem, only. Therefore, this work reduces the number of variables to be computed, which makes numerical treatment of the infinite and semi-infinite time-dependent problems easy.

### Constitutive equations

Using two fundamental elements of elastic spring and viscous dashpot to model time-dependent behavior, construction of the viscoelastic models by suitable combinations of this pair of elements becomes easy. A response closer to that of a real structure is obtained by adding a linear spring in series with the Kelvin solid unit (Mase and Mase 1999). This model is believed to represent SLS, which considers both instantaneous and time-dependent behavior of a specific FG structure. In general, the stress components of elastic and Kelvin viscoelastic parts of the model can be expressed as

$$\sigma_{ij} = \sigma_{ij}^{e_2} = \sigma_{ij}^{K} \tag{1}$$

where $\sigma$ is the total stress and $\sigma^{e2}$ and $\sigma^{K}$ are the elastic and the Kelvin viscoelastic parts of the SLS, respectively. The total strain component can also be divided into two parts as

$$\varepsilon_{ij} = \varepsilon_{ij}^{e_2} + \varepsilon_{ij}^{K} \tag{2}$$

where the Cartesian coordinates are represented by subscripts $i$ and $j$. Also, the Kelvin viscoelastic stress components can be expressed as

$$\sigma_{ij}^{K} = \sigma_{ij}^{\mathrm{el}} + \sigma_{ij}^{\nu} \tag{3}$$

where $\sigma^{\nu}$ and $\sigma^{\mathrm{el}}$ are the viscous and elastic parts of the stresses developed in the Kelvin unit of the SLS model, respectively.

The constitutive equations for the linear elastic part of the SLS model, by assuming infinitesimal strains, can be written according to the generalized Hooke's law as (Mase and Mase 1999):

$$\sigma_{ij}^{e_2} = C_{ijkm}(\boldsymbol{x})\ \varepsilon_{km}^{e_2} \tag{4}$$

where $C_{ijkm}$ are the nonhomogeneous isotropic elasticity tensor, which is a function of the coordinate vector $\boldsymbol{x}$. The most general form of the fourth-order isotropic tensor $C_{ijkm}$ can be shown to have the following form

$$C_{ijkm}(x) = \lambda(x)\,\delta_{ij}\,\delta_{km} + \mu(x)\left(\delta_{ik}\,\delta_{jm} + \delta_{im}\delta_{jk}\right) \tag{5}$$

in which $\lambda$ and $\mu$ are Lame's constants, given by

$$\lambda = \frac{\nu\ E(x)}{(1+\nu)\,(1-2\nu)}, \mu = \frac{E(x)}{2\,(1+\nu)} \tag{6}$$

where $E$ and $\nu$ are Young's modulus and Poisson's ratio, respectively. The exponential material gradation of Lame's constants in the $x$-direction is chosen as

$$\begin{aligned} \mu(x) &= \mu_0\ \ \exp(2\gamma x) \\ \lambda(x) &= \lambda_0\ \ \exp(2\gamma x) \end{aligned} \tag{7}$$

where $\gamma$ is the gradient parameter. A material with exponentially gradation has been widely used in the literature, since such a graded composition represents a justifiable model for certain real situations of the time-dependent nonhomogeneous materials (Jin 2006).

Similarly, the elastic part of the Kelvin viscoelastic stress can be written as

$$\sigma_{ij}^{e_1} = C_{ijkm}(\boldsymbol{x})\,\varepsilon_{km}^{K} = E_1(\boldsymbol{x})\,\bar{C}_{ijkm}\,\varepsilon_{km}^{K} \tag{8}$$

Based on the generalized Newton's law, and using a similar procedure, the viscous stress components of the Kelvin viscoelastic unit can be obtained as (Mase and Mase 1999)

$$\sigma_{ij}^{\nu} = K_{ijmn}(\boldsymbol{x})\,\dot{\varepsilon}_{mn}^{K} \tag{9}$$

where $K_{ijmn}$ represents the nonhomogeneous, isotropic viscosity characteristic tensor of the material that is a function of the spatial variable $x$ and can be defined as

$$\begin{aligned} K_{ijmn}(x) = \beta_\lambda \lambda(x)\ \ \delta_{ij}\delta_{mn} \\ + \beta_\mu \mu(x)\ \ \left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) \end{aligned} \tag{10}$$

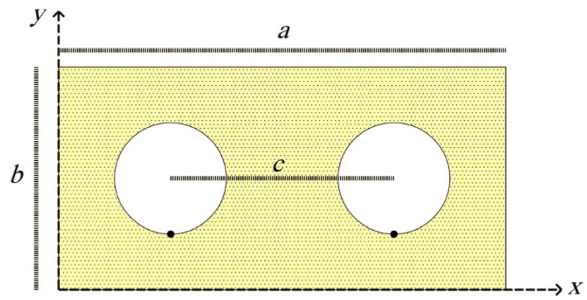


**Figure 1 An FG viscoelastic plate with two circular cutouts under edge tractions at $x = a$.**

in which, $\beta_\mu$ and $\beta_\lambda$ are the hydrostatic and the deviatoric viscosity coefficients of the model, respectively.

Now, based on Eqs. (3) to (10), we derive

$$\sigma_{ij} = E_1(x)\ \bar{C}_{kmij}\,\varepsilon_{km} + E_1(x)\ \bar{K}_{kmij}\,\dot{\varepsilon}_{km} - \left(E_1(x)\ \bar{C}_{kmij}\,\frac{\sigma_{km}}{E_2\,\bar{C}_{kmij}} + E_1(x)\ \bar{K}_{kmij}\,\frac{\dot{\sigma}_{km}}{E_2\,\bar{C}_{kmij}}\right) \tag{11}$$

In obtaining the boundary integral equations, a simplification is employed only for the viscosity coefficients, i.e. $\beta_\mu = \beta_\lambda$. Thus, the general viscoelastic equation of the SLS model can be obtained as

$$\sigma_{ij} = \left(\frac{E_1 E_2}{E_1 + E_2}\right)\ \bar{C}_{ijkm}\,\varepsilon_{km} + \left(\frac{\beta E_2 E_1}{E_1 + E_2}\right)\bar{C}_{ijkm}\,\dot{\varepsilon}_{km} - \left(\frac{\beta E_1}{E_1 + E_2}\right)\dot{\sigma}_{ij} \tag{12}$$

## Boundary integral equations

The viscoelastic integral equations of the boundary and interior points are obtained by imposing the weighted residual technique on the equilibrium equations. In the BEM, the Kelvin fundamental solution of an elastic infinite body is adopted as a proper function for weighting of the differential equilibrium relations as (Sutradhar et al. 2008):

$$0 = \int_D \psi_i\ \left\langle \sigma_{ij,j} + B_i \right\rangle \mathrm{dv} \tag{13}$$

where $B_j$ is a body force acting in the $j$ direction and $\psi_i$ is the fundamental solution which represents effect of a unit concentrated load applied at a point of an infinite domain. After using the fundamental solution in the corresponding Green's function, one can reduce it to the BEM. The direct integral equation requires a displacement fundamental solution. The fundamental solution for 2D elastic problems in exponentially graded structures has been recently derived as (Chan et al. 2004)

$$\psi_i = e^{-2\gamma x}\{C_0 K_0(|\gamma||r|) + C_1 K_1(|\gamma||r|) + G^{ns}\} \tag{14}$$

where

$$\begin{aligned} C_0 &= \frac{3-4\nu}{8\pi\mu_0(1-\nu)}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \\ C_1 &= \frac{\frac{r^2}{|r|}|\gamma| - \frac{\gamma^2}{|\gamma|}|r|}{16\pi\mu_0(1-\nu)}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \end{aligned} \tag{15}$$

in which $K_0$ and $K_1$ are the modified Bessel functions.

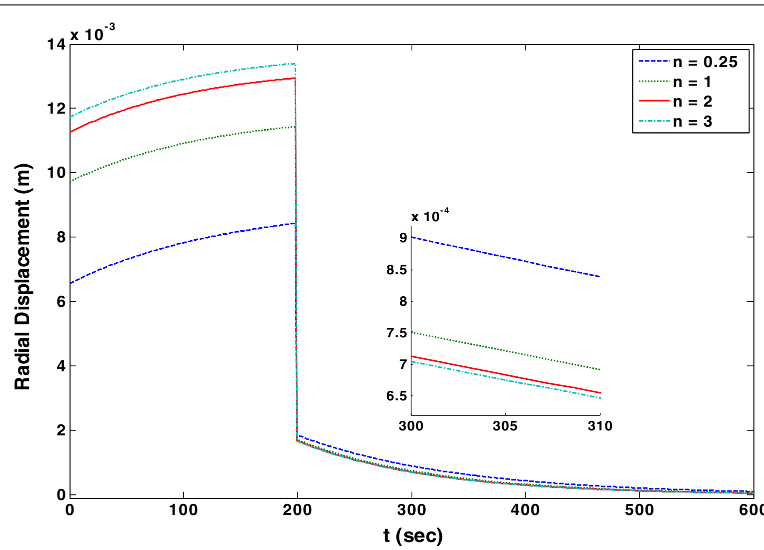


**Figure 2 Effects of the heterogeneity parameter on time history of radial displacement of the viscoelastic plate.** (x=0.75, y=0.5).

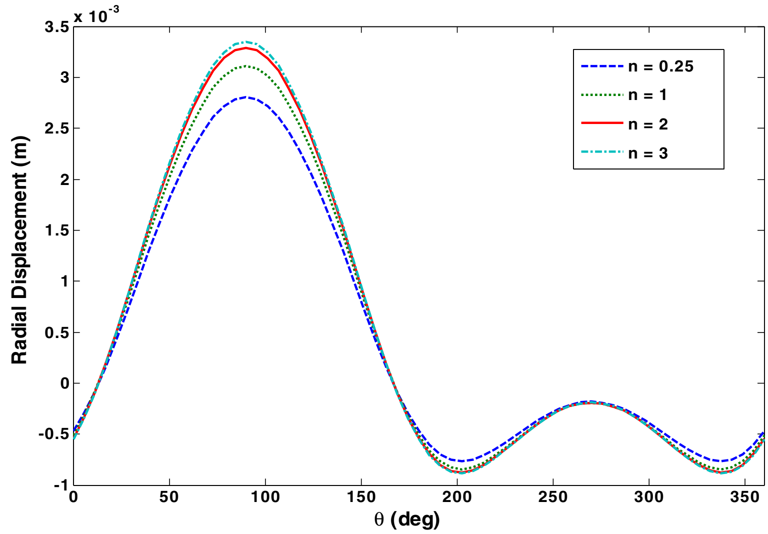


**Figure 3** **Effects of the heterogeneity parameter on time history of radial displacement of the viscoelastic plate.**

By integrating Equation 13 by parts, applying the divergence theorem, and noting that

$$\psi_{i,j} = \frac{1}{2}\left(\psi_{i,j} + \psi_{j,i}\right) = \varepsilon^{\psi}_{ij} \tag{16}$$

in which $\partial D$ is the boundary of the problem and $n_j$ is the outward normal vector; we have

$$0 = \int_{\partial D} \psi_i t_i ds - \int_D \varepsilon^{\psi}_{ij} \sigma_{ij} dv + \int_D \psi_i B_i dv \tag{17}$$

By substituting the viscoelastic constitutive Equation 12 into the integral Equation 17, we have

$$\begin{aligned} 0 = &\int_{\partial D} \psi_i t_i ds - \left(\frac{E_1 E_2}{E_1 + E_2}\right) \int_D \varepsilon^{\psi}_{ij} \left(\bar{C}_{ijmn} \varepsilon_{mn}\right) dv \\ &- \left(\frac{E_1 E_2}{E_1 + E_2}\right) \int_D \varepsilon^{\psi}_{ij} \left(\beta \bar{C}_{ijmn} \dot{\varepsilon}_{mn}\right) dv \\ &+ \left(\frac{E_1}{E_1 + E_2}\right) \int_D \varepsilon^{\psi}_{ij} \left(\beta \dot{\sigma}_{ij}\right) dv + \int_D \psi_i B_i dv \end{aligned} \tag{18}$$

In addition, by taking the fundamental equilibrium equation into account, we have

$$\begin{aligned} &B^{\psi}_i(z) = \Delta(x,z) \quad e_i \\ &\sigma^{\psi}_{ij,j} = -B^{\psi}_i \end{aligned} \tag{19}$$

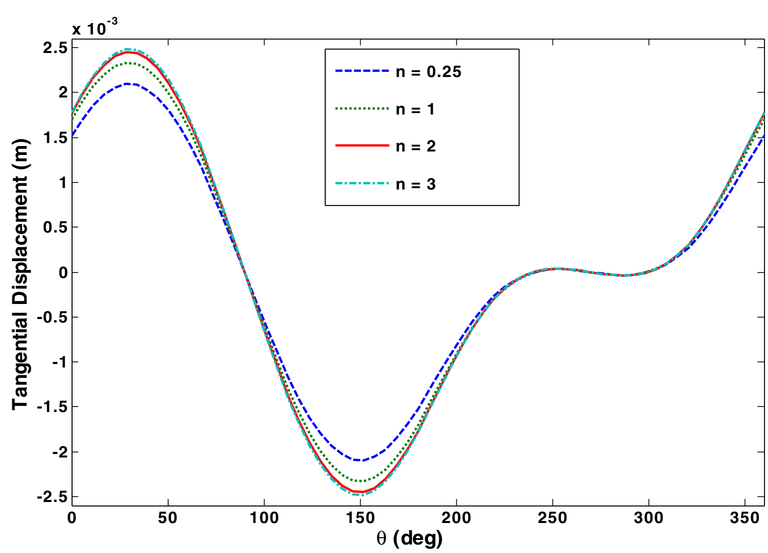


**Figure 4** **Effects of the heterogeneity on tangential displacements of the first hole of the viscoelastic plate.**

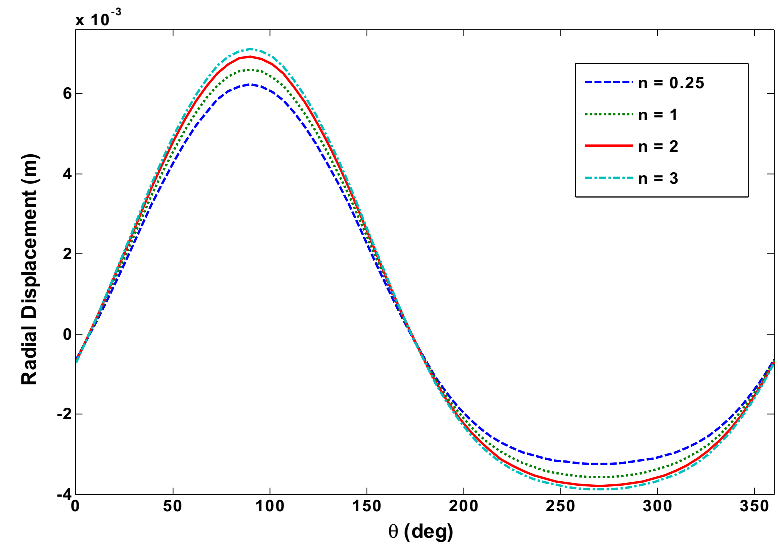


**Figure 5 Effects of the heterogeneity on radial displacements of the second hole of the viscoelastic plate.**

where the unit vector component $e_i$ corresponds to a unit positive force in the $i$ direction applied at $z$ point, and $\Delta\ (x, z)$ is the Dirac delta function, in wherein $z$ and $x$ represent the field and the source points, respectively. Equation 18, by performing some mathematical manipulations, can be rewritten as

$$C_{ji}\,u_i(x) + \beta C_{ji}\dot{u}_i(x) = +\beta\left[\int_{\partial D}\psi_{ji}\,\dot{t}_i ds + \int_D \psi_{ji}\,\dot{B}_i\,dv\right] - \left[\int_{\partial D} t_{ji}^{\psi}\,u_i ds + \beta\int_{\partial D} t_{ji}^{\psi}\,\dot{u}_i ds\right] + \left(\frac{E_1+E_2}{E_1}\right) \times \int_{\partial D}\psi_{ji}\,t_i\,ds + \left(\frac{E_1+E_2}{E_1}\right)\int_D \psi_{ji} B_i\,dv \tag{20}$$

where free term ($\epsilon_{ji}$) is exactly the well-known one of the in elastostatic formulations. Equation 20 may be considered as the integral constitutive equation of time-dependent functionally graded structures modeled properly by the SLS model.

The domain integral of the body forces can easily be transformed into an equivalent boundary integral equation, which results in equation in terms of the boundary values only; however, for more simplicity, we neglect it. In a similar manner, the boundary integral equations of the stresses can be derived.

The next step in numerical discretization is dividing the boundary $\partial D$ into $N_e$ elements; so that after choosing identical numbers of the source points and the nodes, and

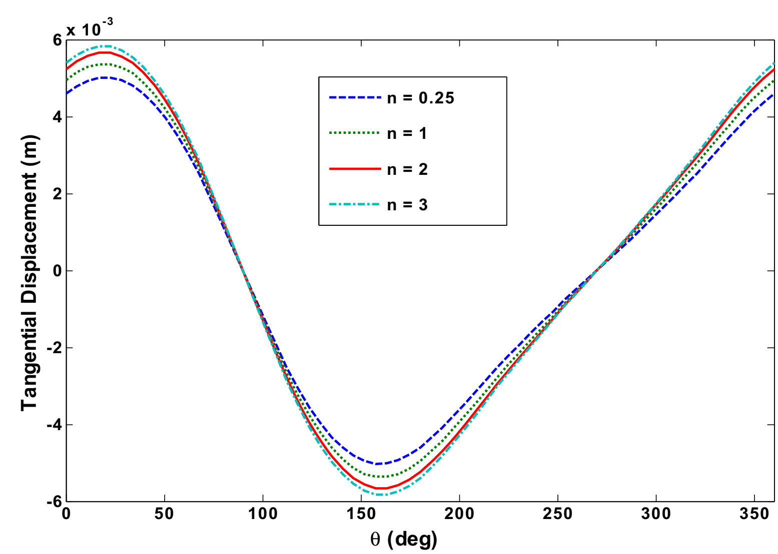


**Figure 6 Effects of the heterogeneity on tangential displacements of the second hole of the viscoelastic plate.**

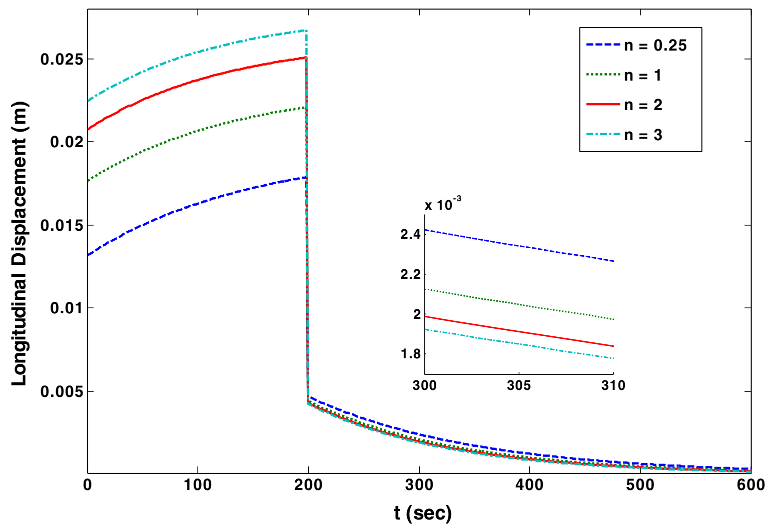


**Figure 7 Effects of the heterogeneity parameter on time history of longitudinal displacement of the viscoelastic plate.** (x=2, y=0).

calculating all integrals, the BEM discretization equations may be derived in a matrix form as follows

$$\mathbf{H}\,\mathbf{u}(t)+\beta\mathbf{H}\,\dot{\mathbf{u}}(t)=\left(\frac{E_1+E_2}{E_1}\right)\mathbf{G}\,\mathbf{t}(t)+(\beta)\,\mathbf{G}\,\dot{\mathbf{t}}(t)+\left(\frac{E_1+E_2}{E_1}\right)\mathbf{D}\,\mathbf{B}(t) \tag{21}$$

It should be noted in the, for solving the above time-dependent differential equation, it was necessary to approximate the displacement and the traction rates in the time domain by a time marching treatment. Finally, the presented algorithm has been cast into a unique program and then solved by the MATLAB software.

## Results and discussion

For evaluating the accuracy and the efficiency of the proposed approach, a numerical viscoelastic problem wherein the material properties are assumed to be exponential functions of the Cartesian coordinate $x$, is considered.

In this regard, a heterogeneous FG viscoelastic plate with exponential material gradation in the $x$-direction and two circular cutouts as shown in Figure 1 is considered. The considered plate is subjected to a uniform traction $P$ in the $x$-direction at the edge $x = a$, where

$$P=\begin{cases}20\,\mathrm{Mpa} & t\le 200\,(s)\\ 0 & 200<t<700\,(s)\end{cases} \tag{22}$$

The opposite end of the considered plate is fixed, while its other boundaries are free of tractions. It is assumed that the plate is sufficiently thin such that a plane stress condition holds. In this problem, Boltzmann's solid model is employed to formulate the differential constitutive equations of the heterogeneous viscoelastic structures.

This problem is solved using the proposed method, adopting the geometric and material parameters as $a = 2$ m, $b = 1$ m, $c = 1$ m, $\gamma = 0.25$, 1, 2 and 3, $\lambda_0 = 5$ GPa, $\mu_0 =$ 350 MPa, $\upsilon = 0.4$ and $\beta = 24$. The diameter of the circular cutouts is $d = 0.35$ m. Due to using the present-graded boundary integral equation approach, only about 54 boundary elements have been used for the boundary discretization. Using the traditional finite element procedures necessitates using thousands of elements and nodal points.

Time history of the radial displacement component of a node of the heterogeneous plate located at ($x = 0.75$, $y = 0.5$) is depicted in Figure 2 for exponential material properties variations and various material gradation exponents.

Time histories of the radial and tangential displacement components of the nodes located at the first hole of the FG viscoelastic plate are shown in Figures 3 and 4 for various gradation exponents, respectively. In addition, the radial and tangential displacement components of the nodes located at the second hole of the FG plate are shown respectively in Figures 5 and 6 for various gradation exponents.

Time history of the longitudinal displacement component of the node ($x = 2$, $y = 0$) of the heterogeneous viscoelastic plate with exponential material properties variations is shown in Figure 7. These results are given for different material gradation exponents.

## Conclusions

In the present paper, a new numerical formulation is presented for accomplishment of the simplified viscoelastic analysis of the functionally graded media by the BEM.

This approach avoids using relaxation functions or mathematical transformations, and it is capable of solving the quasistatic viscoelastic problems with arbitrary load time dependence and arbitrary boundary conditions. A numerical example, as an application, was provided to evaluate this boundary element formulation. Only the displacement and traction fundamental solutions of the FG elastostatic structures are needed in the present formulation. This computational system was easily solved by adopting a linear time approximation for the displacement and traction rates. Other advantage of the presented approach is that the mathematical integral representation needs only the boundary data. Numerical discretization was done without any domain approximations, and the integral equations were applied only on the boundaries.

### Competing interests
The authors declare that they have no competing interests.

### Authors' contributions
HA carried out majority of the activities of the research, including development of the the boundary element solution. MS participated in checking the concepts and preparing the paper. All authors read and approved the final manuscript.

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